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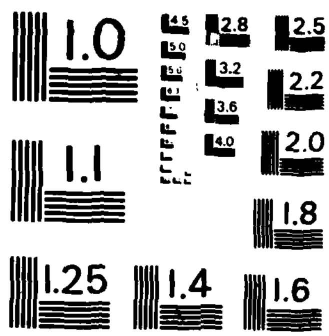
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# ON THE EXTREMUM OF BILINEAR FUNCTIONAL FOR HYPERBOLIC TYPE PARTIAL DIFFERENTIAL EQUATIONS

AD-A142 505

C. N. SHEN

APRIL 1984

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT CENTER
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Transient solutions of the hyperbolic type partial differential equations are needed for solving many engineering problems such as computing stress waves for gun dynamics or determining shock behaviors of penetration mechanics.

Variational procedures using the bilinear formulations with adjoint variables can serve as the theoretical basis in the derivation of algorithms for the

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20. ABSTRACT (CONT'D)

finite element methods, giving direct numerical solutions for partial derivatives of the functions to be found for these problems. The adjoint system can be arranged in a manner that it is a reflected mirror of the original system in time. Generalized boundary conditions employ many types of "springs" relating the various spatial partial derivatives. They are defined to satisfy the boundaries of the concomitant for the bilinear expression. Algorithms for use in the finite element method are simplified since the adjoint system gives exactly the same solutions as that of the original system. The second necessary condition for an extremum is satisfied by showing that the second variation is positive semi-definite.

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## INTRODUCTION

Transient solutions of the hyperbolic type partial differential equation, for example the wave equation or the beam equation, are important for solving engineering problems such as stress wave for gun dynamics or shock behavior of penetration mechanics. At present these equations are usually solved numerically by the finite difference method or by the Galerkin method. Considerable advantage may be obtained if the finite element method can be directly employed instead. Variational procedures using bilinear formulation with adjoint variables can serve as the theoretical basis for the derivation of algorithms using the finite element method for the hyperbolic type partial differential equations (PDE).

## THE VARIATIONAL PRINCIPLE

A dynamical system can be modeled by the following partial differential equation.

$$L(\zeta)\ y(\zeta) = -Q(\zeta) \tag{1}$$

with appropriate boundary and initial conditions. In the above equation L is a linear operator in both spatial and temporal domain, y is the dependent variable, Q is a forcing function, and $\zeta$ represents all independent variables, both spatial and temporal.

The inner product $\langle\ \rangle$ of an adjoint forcing function $\bar{Q}$ and the solution $(y(\zeta))$ of Eq. (1) can be used for the purpose of estimation. This inner product is $\langle\bar{Q},y\rangle$.

An accurate estimation can be made by constructing a variational principle (ref 1). By using the adjoint variable y as a Lagrange multiplier for Eq. (1) adding to $\langle Q,\bar{y}\rangle$, we have

$$J_1[y,\bar{y}] \overset{\Delta}{=} \langle \bar{Q},y\rangle + \langle \bar{y},(Q+Ly)\rangle = \langle \bar{Q},y\rangle + \langle \bar{y},Q\rangle + \langle \bar{y},Ly\rangle \tag{2}$$

To keep the system symmetrical, let us define the adjoint system as

$$\bar{L}(\xi)\bar{y}(\xi) = -\bar{Q}(\xi) \tag{3}$$

By using the original variable y as a Lagrange multiplier for Eq. (3) adding to $\langle Q,\bar{y}\rangle$, we have

$$J_2[y,\bar{y}] \overset{\Delta}{=} \langle Q,\bar{y}\rangle + \langle y,(\bar{Q}+\bar{L}\bar{y})\rangle = \langle Q,\bar{y}\rangle + \langle y,\bar{Q}\rangle + \langle y,\bar{L}\bar{y}\rangle \tag{4}$$

By definition, the relationship of the adjoint system to the original system is

$$D \overset{\Delta}{=} \langle \bar{y},Ly\rangle - \langle y,\bar{L}\bar{y}\rangle = 0 \tag{5}$$

where D is the bilinear concomitant (ref 1). Combining Eqs. (2), (4), and (5) one obtains

$$J_1 = J_2 \tag{6a}$$

In order to keep the functional symmetrical, we have

$$J \overset{\Delta}{=} \frac{1}{2}\,[J_1 + J_2] \tag{6b}$$

which is of the form

$$J = \langle \bar{Q},y\rangle + \langle \bar{y},Q\rangle + \frac{1}{2}\langle \bar{y},Ly\rangle + \frac{1}{2}\langle y,\bar{L}\bar{y}\rangle \tag{6c}$$

To show that the above functional satisfies both the original and the adjoint systems, let us take the first variations of Eqs. (5) and (6) which gives

$$\delta J = \delta J(\delta\bar{y}) + \delta J(\delta y) \tag{7a}$$

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[1] Stacey, Weston, M. Jr., Variational Methods in Nuclear Reactor Physics, Academic Press, 1974.

where

$$\delta J(\delta\bar{y}) = \langle \delta\bar{y},Q\rangle + \frac{1}{2}\langle \delta\bar{y},Ly\rangle + \frac{1}{2}\langle y,\bar{L}\delta\bar{y}\rangle = 0 \tag{7b}$$

and

$$\delta J(\partial y) = \langle \delta y,\bar{Q}\rangle + \frac{1}{2}\langle \delta y,\bar{L}\bar{y}\rangle + \frac{1}{2}\langle \bar{y},L\delta y\rangle = 0 \tag{7c}$$

Also

$$\delta D = \delta D(\delta\bar{y}) + \delta D(\delta y) \tag{8a}$$

where

$$\delta D(\delta\bar{y}) = \langle \delta\bar{y},Ly\rangle - \langle y,\bar{L}\delta\bar{y}\rangle = 0 \tag{8b}$$

and

$$\delta D(\delta y) = -\langle \delta y,\bar{L}\bar{y}\rangle + \langle \bar{y},L\delta y\rangle = 0 \tag{8c}$$

From Eqs. (7b) and (8b) we obtained

$$\delta J(\delta\bar{y}) = \langle \delta\bar{y},Q) + \frac{1}{2}\langle \delta\bar{y},Ly\rangle + \frac{1}{2}\langle \delta\bar{y},Ly\rangle = \langle \delta\bar{y},(Q+Ly)\rangle = 0 \tag{9}$$

For arbitrary $\delta\bar{y}$ satisfying certain general limitations on the boundaries it can be shown that the Euler-Lagrange Equation for the original system in Eq. (1) is satisfied. From Eqs. (7c) and (8c) we get

$$\delta J(\delta y) = \langle \delta y,\bar{Q}\rangle - \frac{1}{2}\langle \delta y,\bar{L}\bar{y}\rangle + \frac{1}{2}\langle \delta y,\bar{L}\bar{y}\rangle = (\delta y,(\bar{Q}+\bar{L}\bar{y})\rangle = 0 \tag{10}$$

For arbitrary variation $\delta y$, the Euler-Lagrange Equation for the adjoint system in Eq. (3) is also satisfied.

## INTEGRAL OF BILINEAR EXPRESSION

The integral of a bilinear expression for a two-dimensional problem having second order partial derivatives in time and fourth order partial derivatives in space can be written as

$$I = \int_{x_o}^{x_b}\int_{t_o}^{t_b} \Omega[y(x,t),\bar{y}(x,t)]dtdx \tag{11}$$

where $\Omega[y,\bar{y}]$ is a given bilinear expression in the form

$$\Omega[y,\bar{y}] = y_t\bar{y}_t - \omega^2 y\bar{y} - a^2 y_x\bar{y}_x - b^2 y_{xx}\bar{y}_{xx} \tag{12}$$

The subscripts t and x indicate the partial derivatives for the functions y and $\bar{y}$.

Equation (12) can be integrated by parts. Two different forms of integration and end conditions are given. The first form of the integral is obtained by integrating by parts on the adjoint variable.

$$I_1 = -\int_{t_o}^{t_b}\int_{x_o}^{x_b} \bar{y}Lydtdx + \int_{x_o}^{x_b} y_t\bar{y}\Big|_{t_o}^{t_b}dx$$

$$+\int_{t_o}^{t_b} \{-b^2y_{xx}\bar{y}_x\Big|_{x_o}^{x_b} + (b^2y_{xx})_x\bar{y}\Big|_{x_o}^{x_b} - a^2y_x\bar{y}\Big|_{x_o}^{x_b}\}dt \tag{13a}$$

On the other hand, we can perform integration on the original variable to give

$$I_2 = -\int_{t_o}^{t_b}\int_{x_o}^{x_b} y\bar{L}\bar{y}dtdx + \int_{x_o}^{x_b} \bar{y}_t y\Big|_{t_o}^{t_b}dx$$

$$+\int_{t_o}^{t_b} \{-b^2\bar{y}_{xx}y_x\Big|_{x_o}^{x_b} + b^2(\bar{y}_{xx})_x y\Big|_{x_o}^{x_b} - a^2\bar{y}_x y\Big|_{x_o}^{x_b}\}dt \tag{13b}$$

To keep the form symmetrical, we take the average of the above two expressions

$$I = \frac{1}{2}I_1 + \frac{1}{2}I_2 = -\int_{x_o}^{x_b}\int_{t_o}^{t_b} \frac{1}{2}(\bar{y}Ly+y\bar{L}\bar{y})dtdx + \frac{1}{2}\int_{x_o}^{x_b} (y_t\bar{y}+\bar{y}_t y)\Big|_{t_o}^{t_b} dx$$

$$+\frac{1}{2}\int_{t_o}^{t_b} (-a^2)(y_x\bar{y}+\bar{y}_x y)\Big|_{x_o}^{x_b} dt$$

$$+\frac{1}{2}\int_{t_o}^{t_b} (-b^2)(y_{xx}\bar{y}_x+\bar{y}_{xx}y_x)\Big|_{x_o}^{x_b} dt - \frac{1}{2}\int_{t_o}^{t_b} [(-b^2y_{xx})_x\bar{y} + (-b^2\bar{y}xx)_x y]_{x_o}^{x_b} dt \tag{14}$$

where

$$Ly = y_{tt} + \omega^2 y - a^2 y_{xx} + b^2 y_{xxxx} \tag{15a}$$

and

$$\overline{Ly} = \bar{y}_{tt} + \omega^2 \bar{y} - a^2 \bar{y}_{xx} + b^2 \bar{y}_{xxxx} \tag{15b}$$

For a fourth order spatial partial and a second order temporal partial system Eq. (5) becomes

$$D = \int_{x_o}^{x_b} \int_{t_o}^{t_b} \bar{y} L y \, dt dx - \int_{x_o}^{x_b} \int_{t_o}^{t_b} y \overline{Ly} \, dt dx \tag{16a}$$

By equating Eqs. (13a) and (13b) and solving for D in Eq. (16a) we are converting the double integral into single integrals in terms of the boundary conditions.

We can express the quantity D as the sum of three parts for end conditions $D_1$, $D_2$, and $D_3$ as

$$D = D_1 + D_2 + D_3 \tag{16b}$$

The terms in $D_1$ involve the initial conditions of y and $\bar{y}$ as

$$D_1 = \int_{x_o}^{x_b} \{ y_t \bar{y} \Big|_{t_o}^{t_b} - \bar{y}_t y \Big|_{t_o}^{t_b} \} dx$$

$$D_1 = \int_{x_o}^{x_b} dx \{ [y_t(x,t_b)\bar{y}(x,t_b) - \bar{y}_t(x,t_b)y(x,t_b)]$$

$$- [\bar{y}_t(x,t_o)\bar{y}(x,t_o) - \bar{y}_t(x,t_o)y(x,t_o)] \} \tag{17a}$$

The terms in $D_2$ involve the boundary conditions from the second partials of y and $\bar{y}$ as

$$D_2 = \int_{t_o}^{t_b} (-a^2) \{ y_x \bar{y} \Big|_{x_o}^{x_b} - \bar{y}_x y \Big|_{x_o}^{x_b} \} dt$$

$$D_2 = \int_{t_o}^{t_b} dt \{ -a^2 [y_x(x_b,t)\bar{y}(x_b,t) - \bar{y}_x(x_b,t)y(x_b,t)$$

$$+ a^2 [y_x(x_o,t)\bar{y}(x_o,t) - \bar{y}_x(x_o,t)y(x_o,t)] \} \tag{17b}$$

The terms in $D_3$ involve the boundary conditions from the fourth partials of $y$ and $\bar{y}$ as

$$D_3 = \int_{t_o}^{t_b} \{-b^2 y_{xx}\bar{y}_x\Big|_{x_o}^{x_b} + b^2 \bar{y}_{xx} y_x\Big|_{x_o}^{x_b} + (b^2 y_{xx})_x \bar{y}\Big|_{x_o}^{x_b} + (-b^2 \bar{y}_{xx})_x y\Big|_{x_o}^{x_b}\} dt$$

$$\begin{aligned} D_3 = \int_{t_o}^{t_b} dt \{ &-b^2[y_{xx}(x_b,t)\bar{y}_x(x_b,t) - \bar{y}_{xx}(x_b,t)y_x(x_b,t)] \\ &+ b^2[y_{xx}(x_o,t)\bar{y}_x(x_o,t) - \bar{y}_{xx}(x_o,t)y_x(x_o,t)]\} \\ + \int_{t_o}^{t_b} dt\{ &-b^2[-y_{xxx}(x_b,t)\bar{y}(x_b,t) + \bar{y}_{xxx}(x_b,t)y(x_b,t)] \\ &+ b^2[-y_{xxx}(x_o,t)\bar{y}(x_o,t) + \bar{y}_x(x_o,t)y(x_o,t)]\} \end{aligned} \tag{17c}$$

In order that $D \equiv 0$ in Eq. (16b) it is sufficient that

$$D_1 \equiv 0 \tag{18a}$$

$$D_2 \equiv 0 \tag{18b}$$

and

$$D_3 \equiv 0 \tag{18c}$$

## THE SYMMETRICAL ADJOINT SYSTEM

The adjoint independent variable $\tau$ in Figure 1 can be expressed as

$$\frac{\tau_b - \tau}{\tau_b - \tau_o} = \frac{t - t_o}{t_b - t_o} \tag{19}$$

which gives

$$\tau = \tau_b \quad \text{for} \quad t = t_o \tag{20a}$$

and

$$\tau = \tau_o \quad \text{for} \quad t = t_b \tag{20b}$$

It is noted from Eq. (19) that

$$\tau_b - \tau_o = t_b - t_o \tag{21a}$$

$$\tau = \tau_b + t_o - t \tag{21b}$$

$$d\tau = -dt \tag{21c}$$

$$\frac{d}{d\tau} = -\frac{d}{dt} \tag{21d}$$

and

$$\bar{y}(x,t) = y(x,\tau = \tau_b+t_o-t) \tag{21e}$$

Let us assume that the adjoint system shown in Figure 1 gives

$$\bar{y}(x,\hat{t}=t) = y(x,\hat{t}=t_b+t_o-t) \tag{22a}$$

$$\bar{y}_t(x,\hat{t}=t) = -y_t(x,\hat{t}=t_b+t_o-t) \tag{22b}$$

$$\bar{y}_x(x,\hat{t}=t) = y_x(x,\hat{t}=t_b+t_o-t) \tag{22c}$$

where $\hat{t}$ is a dummy variable for t.

We may define the adjoint system as the image reflection in the time domain of the original system. Equation (22) yields the following known initial conditions

$$\bar{y}(x,\hat{t}=t_b) = y(x,\hat{t}=t_o) \quad \text{(known)} \tag{23a}$$

$$\bar{y}_t(x,\hat{t}=t_b) = -y_t(x,\hat{t}=t_o) \quad \text{(known)} \tag{23b}$$

The interpretation of the above equations gives the initial conditions of the original system as the far end conditions for the adjoint system, since the adjoint system is a reflected mirror of the original system in time.

## INITIAL CONDITIONS FOR THE ADJOINT SYSTEM

We take a symmetry approach for the initial conditions of the adjoint system as

$$\bar{y}(x,t=t_b) = y(x,t=t_o) \quad , \quad \bar{y}_t(x,t=t_b) = -y_t(x,t=t_o) \tag{24}$$

$$\bar{y}(x,t=t_o) = y(x,t=t_b) \quad , \quad \bar{y}_t(x,t=t_o) = -y_t(x,t=t_b) \tag{25}$$

Thus Eq. (17a) becomes

$$D_1 = \int_{x_o}^{x_b} dx\{[y_t(x,t=t_b)y(x,t=t_o) + y_t(x,t=t_o)y(x,t=t_b)$$

$$- [y_t(x,t=t_o)y(x,t=t_b) + y_t(x,t=t_b)y(x,t=t_o)]\} = 0 \tag{26}$$

Since the integrand of Eq. (26) is zero, the above satisfies Eq. (18a). The initial conditions in Eq. (25) are given. Therefore

$$\delta\bar{y}(x,t=t_b) = \delta y(x,t=t_o) = 0 \tag{27a}$$

$$\delta\bar{y}_t(x,t=t_b) = -\delta y_t(x,t=t_o) = 0 \tag{27b}$$

THE GENERALIZED BOUNDARY CONDITIONS

Let us consider the operator L in Eq. (15a) for two different cases as follows.

A. For the wave equation we have

$$Ly = y_{tt} - a^2 y_{xx} \tag{28}$$

Let us assume that elastic springs are installed at the ends such that

$$y_x(x_b,t) = k_b y(x_b,t) \quad , \quad \bar{y}_x(x_b,t) = k_b \bar{y}(x_b,t) \tag{29a}$$

$$y_x(x_o,t) = -k_o y(x_o,t) \quad , \quad \bar{y}_x(x_o,t) = -k_o \bar{y}(x_o,t) \tag{29b}$$

This is equivalent to state that the fixed end condition for a prismatic bar is $k_b = k_o \to \infty$ and the free end condition is $k_b = k_o \to 0$. If Eq. (29) is substituted into Eq. (17b) we have

$$D_2 = 0 \tag{30}$$

B. For the beam equation we have

$$Ly = y_{tt} + b^2 y_{xxxx} \tag{31}$$

Two sets of springs are incorporated at the ends. They are:

(1) Torsional springs relate the moments (the second partials) with the slopes (the first partials)

$$y_{xx}(x_b,t) = n_b y_x(x_b,t) \qquad \bar{y}_{xx}(x_b,t) = n_b \bar{y}_x(x_b,t) \tag{32a}$$

$$y_{xx}(x_o,t) = -n_o y_x(x_o,t) \qquad \bar{y}_{xx}(x_o,t) = -n_o \bar{y}_x(x_o,t) \tag{32b}$$

(2) Linear springs relate the shears (the third partials) with the deflection (no partials)

$$y_{xxx}(x_b,t) = C_b y(x_b,t) \qquad \bar{y}_{xxx}(x_b,t) = C_b \bar{y}(x_b,t) \tag{33a}$$

$$y_{xxx}(x_o,t) = -C_o y(x_o,t) \qquad \bar{y}_{xxx}(x_o,t) = -C_o \bar{y}(x_o,t) \tag{33b}$$

By substituting Eqs. (32) and (33) into Eq. (17c) we have

$$D_3 = 0 \tag{34}$$

Table I shows the assignment of the spring constants for various physical end conditions.

TABLE I. GENERALIZED BOUNDARY CONDITIONS

| | At Fixed End | At Hinged End | At Guided End | At Free End |
|---|---|---|---|---|
| | $y=\bar{y}=0$ | $y=\bar{y}=0$ | $y_x=\bar{y}_x=0$ | $y_{xx}=\bar{y}_{xx}=0$ |
| | $y_x=\bar{y}_x=0$ | $y_{xx}=\bar{y}_{xx}=0$ | $y_{xxx}=\bar{y}_{xxx}=0$ | $y_{xxx}=\bar{y}_{xxx}=0$ |
| | $\delta y=\delta\bar{y}=0$ | $\delta y=\delta\bar{y}=0$ | $\delta y_x=\delta\bar{y}_x=0$ | $\delta y_{xx}=\delta\bar{y}_{xx}=0$ |
| | $\delta y_x=\delta\bar{y}_x=0$ | $\delta y_{xx}=\delta\bar{y}_{xx}=0$ | $\delta y_{xxx}=\delta\bar{y}_{xxx}=0$ | $\delta y_{xxx}=\delta\bar{y}_{xxx}=0$ |
| Torsional Spring $y_{xx}\overset{\Delta}{=}\eta y_x$ | $\eta \to \infty$ | $\eta \to 0$ | $\eta \to \infty$ | $\eta \to 0$ |
| Deflection Spring $y_{xxx}=cy$ | $c \to \infty$ | $c \to \infty$ | $c \to 0$ | $c \to 0$ |
| Spring $y_x=ky$ | $\delta y=\delta\bar{y}=0$<br>$\delta y_x=\delta\bar{y}_x=0$ | $k \to \infty$ | $k \to 0$ | $k$ = undetermined |

## THE FIRST VARIATION

The sum of the two functionals is obtained by adding Eqs. (6c) and (14) as

$$J + I = \int_{x_o}^{x_b} \int_{t_o}^{t_b} (\bar{Q}y+y\bar{Q})dxdt + T + W + B \tag{35}$$

where

$$T = \frac{1}{2} \int_{x_o}^{x_b} (y_t\bar{y}+\bar{y}_t y)\Big|_{t_o}^{t_b} dx \quad , \quad W = \frac{1}{2} \int_{t_o}^{t_b} (-a^2)(y_x\bar{y}+\bar{y}_x y)\Big|_{x_o}^{x_b} dt$$

and

$$B = \frac{1}{2} \int_{t_o}^{t_b} (-b^2)(y_{xx}\bar{y}_x+\bar{y}_{xx}y_x)\Big|_{x_o}^{x_b} dt - \frac{1}{2} \int_{t_o}^{t_b} [(-b^2y_{xx})_x\bar{y} + (-b^2\bar{y}_{xx})_x y\Big|_{x_o}^{x_b} dt \tag{36}$$

By taking the variations $\delta\bar{y}$ and $\delta y$ separately, we let

$$\delta J = \delta J(\delta\bar{y}) + \delta J(\delta y) \tag{37}$$

Then one obtains from Eqs. (35) and (36) that

$$\delta J(\delta\bar{y}) = -\delta I(\delta\bar{y}) + \iint Q\delta\bar{y}\, dxdt + \delta I(\delta\bar{y}) + \delta W(\delta\bar{y}) + \delta B(\delta\bar{y}) = 0$$

where

$$\delta T(\delta\bar{y}) = \frac{1}{2} \int_{x_o}^{x_b} (y_t\delta\bar{y}+y\delta\bar{y}_t)\Big|_{t_o}^{t_b} dx \quad , \quad \delta W(\delta\bar{y}) = \frac{1}{2} \int_{t_o}^{t_b} (-a^2)(y_x\delta\bar{y}+y\delta\bar{y}_x)\Big|_{x_o}^{x_b} dt$$

and

$$\delta B(\delta\bar{y}) = \frac{1}{2} \int_{t_o}^{t_b} (-b^2)(y_{xx}\delta\bar{y}_x+y_x\delta\bar{y}_{xx})\Big|_{x_o}^{x_b} dt$$

$$- \frac{1}{2} \int_{t_o}^{t_b} [(-b^2)y_{xxx}\delta\bar{y} + (-b^2)y\delta\bar{y}_{xxx}]_{x_o}^{x_b} dt \tag{38}$$

where $-\delta I(\delta\bar{y})$ can be derived from Eqs. (11) and (12) as

$$-\delta I(\delta\bar{y}) = -\int_{x_o}^{x_b} \int_{t_o}^{t_b} (y_t\delta\bar{y}_t-\omega^2y\delta\bar{y}-a^2y_x\delta\bar{y}_x-b^2y_{xx}\delta\bar{y}_{xx})dxdt \tag{39}$$

The second term on the right side of Eq. (37) is

$$\delta J(\delta y) = -\delta I(\delta y) + \iint \bar{Q}\delta y\, dxdt + \delta T(\delta y) + \delta W(\delta y) + \delta B(\delta y) = 0$$

where

$$\delta T(\delta y) = \frac{1}{2}\int_{x_o}^{x_b} (\bar{y}_t\delta y+\bar{y}\delta y_t)\Big|_{t_o}^{t_b} dx \quad , \quad \delta W(\delta y) = \frac{1}{2}\int_{t_o}^{t_b} (-a^2)(\bar{y}_x\delta y+\bar{y}\delta y_x)\Big|_{x_o}^{x_b} dt$$

$$\delta B(\delta y) = \frac{1}{2}\int_{t_o}^{t_b} (-b^2)(\bar{y}_{xx}\delta y_x+\bar{y}_x\delta y_{xx})\Big|_{x_o}^{x_b} dt$$

$$- \int_{t_o}^{t_b} [(-b^2\bar{y}_{xxx})\delta y-b^2\bar{y}\delta y_{xxx}]_{x_o}^{x_b} dt \tag{40}$$

and

$$-\delta I(\delta y) = - \int_{x_o}^{x_b}\int_{t_o}^{t_b} (\bar{y}_t\delta y_t-\omega^2\bar{y}\delta y-a^2\bar{y}_x\delta y_x(-b^2)\bar{y}_{xx}\delta y_{xx})dxdt \tag{41}$$

It is noted that Eqs. (38) and (40) are exactly the same form, where Eqs. (39) and (41) are also similar.

For the beam equation it is noted that the high partials in Eqs. (38) and (39) can be replaced by Eqs. (32) and (33). The variations of the adjoint higher partials from these equations can be written as

$$\delta\bar{y}_{xx}(x_b,t) = n_b\delta\bar{y}_x(x_b,t) \quad \delta\bar{y}_{xxx}(x_b,t) = c_b\delta y(x_b,t) \tag{42a}$$

$$\delta\bar{y}_{xx}(x_o,t) = -n_o\delta\bar{y}_x(x_o,t) \quad \delta\bar{y}_{xxx}(x_o,t) = -c_o\delta y(x_o,t) \tag{42b}$$

Equations (38) and (39), with the aid of Eqs. (32), (33), and (42), are the key equations to be used for the finite element method. It is noted that the first variation $\delta J(\delta\bar{y})$ is the same as the first variation $\delta J(\delta y)$ by adding or dropping the bar on top of the variables and their variations. We do not need to solve for the adjoint system in Eqs. (40) and (41) since they give exactly the same solutions as that of the original system.

SECOND VARIATIONS

The functions $y$ and $\bar{y}$ and their partials are written in the form in terms of a small parameter $\mu$

$$y(x,t,\mu) = y(x,t) + \delta y(x,t,\mu) \quad , \quad \delta y(x,t,u) = \mu\eta(x,t) \tag{43a}$$

$$y_t(x,t,\mu) = y_t(x,t) + \delta y_t(x,t,\mu) \quad , \quad \delta y_t(x,t,u) = \mu\eta_t(x,t) \tag{43b}$$

$$y_x(x,t,\mu) = y_x(x,t) + \delta y_x(x,t,\mu) \quad , \quad \delta y_x(x,t,u) = \mu\eta_x(x,t) \tag{43c}$$

$$\bar{y}(x,t,\mu) = \bar{y}(x,t) + \delta\bar{y}(x,t,\mu) \quad , \quad \delta\bar{y}(x,t,u) = \mu\bar{\eta}(x,t) \tag{43d}$$

$$\bar{y}_t(x,t,\mu) = \bar{y}_t(x,t) + \delta\bar{y}_t(x,t,\mu) \quad , \quad \delta\bar{y}_t(x,t,u) = \mu\bar{\eta}_t(x,t) \tag{43e}$$

$$\bar{y}_x(x,t,\mu) = \bar{y}_x(x,t) + \delta\bar{y}_x(x,t,\mu) \quad , \quad \delta\bar{y}_x(x,t,u) = \mu\bar{\eta}_x(x,t) \tag{43f}$$

Similar expressions can be derived for higher partials in x. Thus, the functional $J(\mu)$ can be expressed as (ref 2)

$$J(u) = J(u=0) + \delta J + \delta^2 J \tag{44a}$$

where

$$\delta J = \mu\left(\frac{\partial J}{\partial \mu}\right)_{u=0} \tag{44b}$$

and

$$\delta^2 J = \frac{\mu^2}{2}\left(\frac{\partial^2 J}{\partial \mu^2}\right)_{u=0} \tag{44c}$$

By taking variations of $\delta J(\delta\bar{y})$ in Eqs. (38) and (39) and some for $\delta J(\delta y)$ in Eqs. (40) and (41), we have

$$\delta^2 J = \delta^2 T + \delta^2 B + \delta^2 W - \delta^2 I \tag{45a}$$

where

$$\delta^2 T = \frac{1}{2}\int_{x_b}^{x_o} (\delta y_t \delta\bar{y} + \delta y \delta\bar{y}_t)\Big|_{t_o}^{t_b} dx \tag{45b}$$

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[2]Rund, H., The Hamilton-Jacobi Theory of the Calculus of Variations, Robert E. Krieger Publishing Company, Huntington, NY, 1973.

$$\delta^2 B = \frac{1}{2} \int_{t_o}^{t_b} (-b^2)(\delta y_{xx}\delta\bar{y}_x + \delta y_x \delta\bar{y}_{xx}) \Big|_{x_o}^{x_b} dt$$

$$+ \frac{1}{2} \int_{t_o}^{t_b} b^2(\delta y_{xxx}\delta\bar{y} + \delta y \delta\bar{y}_{xxx}) \Big|_{x_o}^{x_b} dt \qquad (45c)$$

and

$$\delta^2 W = \frac{1}{2} \int_{t_o}^{t_b} (-a^2)(\delta y_x \delta\bar{y} + \delta y \delta\bar{y}_x) \Big|_{x_o}^{x_b} dt \qquad (45d)$$

The second variation of I is obtained from Eqs. (39) and (41) as

$$\delta^2 I = \frac{\mu^2}{2} \left(\frac{\partial^2 I}{\partial \mu^2}\right)_{\mu=0}$$

$$= \frac{1}{2} \delta y[\delta I(\delta y)] + \frac{1}{2} \delta\bar{y}[\delta I(\delta y)]$$

$$= \frac{1}{2} \int_{x_o}^{x_b} \int_{t_o}^{t_b} (\delta y_t \delta\bar{y}_t - \omega^2 \delta y \delta\bar{y} - a^2 \delta y_x \delta\bar{y}_x - b^2 \delta y_{xx} \delta\bar{y}_{xx}) dx dt$$

$$+ \frac{1}{2} \int_{x_o}^{x_b} \int_{t_o}^{t_b} (\delta\bar{y}_t \delta y_t - \omega^2 \delta\bar{y} \delta y - a^2 \delta\bar{y}_x \delta y_x - b^2 \delta\bar{y}_{xx} \delta y_{xx}) dx dt$$

$$\delta^2 I = \int_{x_o}^{x_b} \int_{t_o}^{t_b} (\delta\bar{y}_t \delta y_t - \omega^2 \delta\bar{y} \delta y - a^2 \delta\bar{y}_x \delta y_x - b^2 \delta\bar{y}_{xx} \delta y_{xx}) dx dt \qquad (45e)$$

Substituting Eq. (27) into Eq. (45b) we have

$$\delta^2 T = 0 \qquad (46a)$$

For all the end conditions in Table I either the variations $\delta y_{xx}$ and $\delta\bar{y}_{xx}$ must vanish or $\delta y_x$ and $\delta\bar{y}_x$ must vanish. Thus, the first term on the right side of Eq. (45c) is zero. Similarly, for all the end conditions in Table I either the variations $\delta y_{xx}$ and $\delta\bar{y}_{xx}$ must vanish or $\delta y$ and $\delta\bar{y}$ must vanish. Thus the second term on the right side of Eq. (45c) is also zero. The third term is zero except at the guided end. Thus, in general

$$\delta^2 B = 0 \qquad (46b)$$

In Table I, except the free end, either the $\delta y_x$ and $\delta\bar{y}_x$ must vanish or $\delta y$ and $\delta\bar{y}$ must vanish. Thus, one obtains

$$\delta^2 W = 0 \tag{46c}$$

This reduces the second variations $\delta^2 J$ to

$$\delta^2 J = -\delta^2 J \tag{47}$$

as given in Eq. (45e).

Substituting Eq. (45e) into Eq. (47) gives

$$\delta^2 J = \int_{t_o}^{t_b}\int_{x_o}^{x_b} [-\delta y_t(x,t)\delta\bar{y}_t(x,t) + \omega^2\delta y(x,t)\delta\bar{y}(x,t) + $$

$$+ a^2\delta y_x(x,t)\delta\bar{y}_x(x,t) + b^2\delta y_{xx}(x,t)\delta\bar{y}_{xx}(x,t)]dxdt \tag{48}$$

In order that the functional J is an extremum (refs 3,4), the second variation $\delta^2 J$ must be either positive semi-definite or negative semi-definite, i.e.,

$$\delta^2 J \geq 0 \quad (\text{or } \delta^2 J \leq 0) \tag{49}$$

The above is a necessary condition for a minimum (or a maximum).

The adjoint variations in Eq. (48) may be obtained by the relations given in Eq. (22) as

$$\delta\bar{y}(x,\hat{t}=t) = \delta y(x,\hat{t}=t_b+t_o-t) \tag{50a}$$

$$\delta\bar{y}_t(x,\hat{t}=t) = -\delta y_t(x,\hat{t}=t_b+t_o-t) \tag{50b}$$

$$\delta\bar{y}_x(x,\hat{t}=t) = \delta y_x(x,\hat{t}=t_b+t_o-t) \tag{50c}$$

The variations of adjoint initial conditions can be derived from Eq. (23) as

$$\delta\bar{y}(x,\hat{t}=t_b) = \delta y(x,\hat{t}=t_b) = 0 \quad \text{for all } x \tag{51a}$$

$$\delta\bar{y}_t(x,\hat{t}=t_b) = -\delta y_t(x,t=t_o) = 0 \quad \text{for all } x \tag{51b}$$

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[3]Gelfand, I. M. and Formin, S. V., Calculus of Variations, Prentice-Hall, 1963.

[4]Sagan, Hans, Introduction to the Calculus of Variations, McGraw-Hill, 1969.

By substituting Eq. (51) into Eq. (48), we have

$$\delta^2 J = \int_{t_o}^{t_b} \int_{x_o}^{x_b} P(x,t)dxdt \tag{52a}$$

where

$$P(x,t) = \delta y_t(x,t)\delta y_t(x,t_b+t_o-t) + \omega^2\delta y(x,t)\delta y_x(x,t_b+t_o-t)$$

$$+ a^2\delta y_x(x,t)\delta y_x(x,t_b+t_o-t) + b^2\delta y_{xx}(x,t)\delta y_{xx}(x,t_b+t_o-t) \tag{52b}$$

SENSITIVITY RELATIONSHIP

In order to show that the second variation of the functional J is positive semi-definite, one needs to obtain the variations of the function and its partials together with that of the adjoint functions and its partials as indicated in Eq. (48). We can get these variations through the study of the sensitivity coefficients (ref 5) and its relationship to the parameters given in Eq. (43). Let the forcing function in Eq. (1) be

$$Q(x,t) = qf(x,t) \tag{53}$$

It is assumed that the forcing function parameter q is subject to a small constant perturbation $\delta q$ as

$$q = q_o + \delta q \tag{54}$$

Then the variation of the function y is

$$\delta y(x,t) = \frac{\partial y(x,t)}{\partial q}\delta q = \nu(x,t)\delta q \tag{55a}$$

where

$$\nu(x,t) = \frac{\partial y}{\partial q} \tag{55b}$$

The quantity $\nu$ is the sensitivity coefficient for the variation $\delta y(x,t)$ due to a small constant perturbation $\delta q$.

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[5]Tomovic, Rajko, Sensitivity Analysis of Dynamic Systems, McGraw-Hill, 1963.

The original PDE in Eq. (15a) can be written as

$$\phi = Ly + Q$$

$$= y_{tt} + \omega^2 y - a^2 y_{xx} + b^2 y_{xxxx} + qf(x,t) = 0 \qquad (56)$$

Due to the perturbation of q the change of $\phi$ obeys the following relationship

$$\frac{\partial\phi}{\partial y_{tt}} \frac{\partial y_{tt}}{\partial q} + \frac{\partial\phi}{\partial y_{xx}} \frac{\partial y_{xx}}{\partial q} + \frac{\partial\phi}{\partial y_{xxx}} \frac{\partial y_{xxxx}}{\partial q} + f(x,t) = 0 \qquad (57)$$

It is also noted from Eq. (56) that

$$\frac{\partial\phi}{\partial y_{tt}} = 1 \quad , \quad \frac{\partial\phi}{\partial y} = \omega^2 \qquad (58a)$$

$$\frac{\partial\phi}{\partial y_{xx}} = -a^2 \quad , \quad \frac{\partial\phi}{\partial y_{xxxx}} = b^2 \qquad (58b)$$

Using the definition in Eq. (55b) the partials can be interchanged as

$$\frac{\partial y_{tt}}{\partial q} = \frac{\partial^2}{\partial t^2}\left(\frac{\partial y}{\partial q}\right) = \nu_{tt} \qquad (59a)$$

$$\frac{\partial y_{xx}}{\partial q} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial q}\right) = \nu_{tt} \qquad (59b)$$

and

$$\frac{\partial y_{xxxx}}{\partial q} = \frac{\partial^4}{\partial x^4}\left(\frac{\partial y}{\partial q}\right) = \nu_{xxxx} \qquad (59c)$$

Substituting Eqs. (58) and (59) into Eq. (57) we have

$$\nu_{tt} + \omega^2\nu - a^2\nu_{xx} + b^2\nu_{xxxx} + f(x,t) = 0 \qquad (60)$$

If we compare the definitions of variation in Eq. (43a) with the definition of sensitivity relationship in Eq. (55a) we have

$$\delta y(x,t) = \mu\eta(x,t) = (\delta q)\nu(x,t) \qquad (61)$$

which gives

$$\eta(x,t) = \nu(x,t) \qquad (62a)$$

and

$$\delta q = \mu \tag{62b}$$

Thus Eq. (60) becomes

$$\eta_{tt} + \omega^2\eta - a^2\eta_{xx} + b^2\eta_{xxxx} + f(x,t) = 0 \tag{63}$$

which gives the PDE of the variations of the original system.

If we compare Eq. (63) with Eq. (56) we see that the variation $\eta(x,t) = \mu^{-1}\delta y(x,t)$ in Eq. (63) takes the place of the function y in Eq. (56) with q = 1. Therefore, the PDE for the variations is unchanged except by a scale factor. Thus the solution of the variation $\delta y(x,t)$ has the same form as that of the original function y.

Similarly for the adjoint system one can obtain

$$\delta\bar{y}(x,t) = \overline{\mu\eta}(x,t) = (\delta\bar{q})\bar{\nu}(x,t) \tag{64}$$

$$\bar{\eta}(x,t) = \bar{\nu}(x,t) \tag{65a}$$

$$\delta\bar{q} = \bar{\mu} \tag{65b}$$

and

$$\bar{\eta}_{tt} + \omega^2\bar{\eta} - a^2\bar{\eta}_{xx} + b^2\bar{\eta}_{xxxx} + \bar{f}(x,t) = 0 \tag{66}$$

which is the PDE of the variations of the adjoint system.

## EXTREMAL OF FUNCTIONAL FOR A SIMPLE OSCILLATOR

To show that $\delta^2 J$ must be positive semi-definite we start with an example by a simple harmonic oscillator with no forcing function. Thus from Eq. (63) we have the ordinary differential equation (ref 6)

$$\eta_{tt} + \omega^2\eta = 0 \tag{67}$$

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[6]Shen, C. N. and Wu, Julian J., "A New Variational Method for Initial Value Problems, Using Piecewise Hermite Polynomial Spline Functions," ARO Report 81-3, Proceedings of the 1981 Army Numerical Analysis and Computers Conference, 1981.

The solution for the above equation is

$$\delta y = \mu\eta = A\cos(\omega t+\theta) \tag{68a}$$

$$\delta y_t = \mu\eta_t = -\omega A\sin(\omega t+\theta) \tag{68b}$$

Both A and θ can be determined from the following given initial conditions

$$\delta y(t=0) = \delta y(0) = A\cos\theta \tag{69a}$$

$$\delta y_t(t=0) = \delta yt(0) = -\omega A\sin\theta \tag{69b}$$

For computation by the finite element method the increment time is taken as T which gives

$$T = t_b - t_o = \left(\frac{n}{\omega}\right)\left(\frac{\pi}{2}\right) \tag{70}$$

where $n = 1,2,3...$

The image function becomes

$$\delta y(\hat{t}=T-t) = A\cos[\theta+\omega(T-t)] \tag{71a}$$

$$\delta y_t(\hat{t}=T-t) = -\omega A\sin[\theta+\omega(T-t)] \tag{71b}$$

For the ordinary differential equation we have the second variation from Eq. (52) which gives

$$\delta^2 J = \int_o^T [\delta y_t(x,\hat{t}=t)\delta y_t(x,\hat{t}=T-t)$$

$$+ \omega^2\delta y(x,\hat{t}=t)\delta y(x,\hat{t}=T-t)]dt \tag{72}$$

Separating Eq. (72) into two parts and using Eqs. (68) and (71) we have

$$\delta^2 J = \delta^2 J[\delta y_t] + \delta^2 J[\omega\delta y] \tag{73a}$$

where

$$\delta^2 J[\delta y_t] = \int_o^{\omega T} \omega^2 A^2 \sin(\theta+\omega t)\sin(\theta+\omega T-\omega t)d(\omega t) \tag{73b}$$

$$\delta^2 J[\omega\delta_y] = \int_o^{\omega T} \omega^2 A^2 \cos(\theta+\omega t)\cos(\theta+\omega T-\omega t)d(\omega t) \tag{73c}$$

and

$$\omega T = n(\frac{\pi}{2}) \tag{73d}$$

which is a multiple of $\pi/2$.

The trigonometric relationship for Eq. (73) is

$$\sin(\omega t+\theta) = \sin \omega t \cos \theta + \cos \omega t \sin \theta \tag{74a}$$

$$\cos(\omega t+\theta) = \cos \omega t \cos \theta - \sin \omega t \sin \theta \tag{74b}$$

$$\sin(\theta+\omega T-\omega t) = -\sin[\omega t - (\theta+n\pi/2)]$$

$$= -\sin \omega t \cos(\theta+n\pi/2) + \cos \omega t \sin(\theta+n\pi/2) \tag{74c}$$

and

$$\cos(\theta+\omega T-\omega t) = \cos[\omega t - (\theta+n\pi/2)]$$

$$= \cos \omega t \cos(\theta+n\pi/2) + \sin \omega t \sin(\theta+n\pi/2) \tag{74d}$$

For the case when n is odd, we have

$$\cos(\theta+n\pi/2) = (-1)^{\frac{n+1}{2}} \sin \theta \tag{75a}$$

$$\sin(\theta+n\pi/2) = (-1)^{\frac{n-1}{2}} \cos \theta \tag{75b}$$

*For the case when n is even, we have*

$$\cos(\theta+n\pi/2) = (-1)^{n/2} \cos \theta \tag{76a}$$

$$\sin(\theta+n\pi/2) = (-1)^{n/2} \sin \theta \tag{76b}$$

First, we take the case when n is odd. Substituting Eqs. (74) and (75) into Eq. (73), one obtains

$$\delta^2 J[\delta y_t] = \omega^2 A^2 \int_o^{n\pi/2} \{(\sin \omega t \cos \theta + \cos \omega t \sin \theta)$$

$$\cdot [- \sin \omega t(-1)^{(n+1)/2} \sin \theta + \cos \omega t(-1)^{(n-1)/2} \cos \theta]\} d(\omega t)$$

$$= (-1)^{(n-1)/2} \omega^2 A^2 \int_o^{n\pi/2} [\sin \theta \cos \theta + \sin \omega t \cos \omega t] d(\omega t)$$

$$= (-1)^{(n-1)/2} \omega^2 A^2 [\frac{1}{2} + \frac{n\pi}{2} \sin \theta \cos \theta] \tag{77a}$$

and

$$\delta^2 J[\omega\delta y] = \omega^2 A^2 \int_o^{n\pi/2} \{(\cos\omega t \cos\theta - \sin\omega t \sin\theta)$$

$$\cdot [\cos\omega t(-1)^{(n+1)/2} \sin\theta + \sin\omega t\, (-1)^{(n-1)/2} \cos\theta]\} d(\omega t)$$

$$= (-1)^{(n-1)/2}\, \omega^2 A^2 \int_o^{n\pi/2} [-\sin\theta\cos\theta + \sin\omega t \cos\omega t] d(\omega t)$$

$$= (-1)^{(n-1)/2}\, \omega^2 A^2\, [\frac{1}{2} - \frac{n\pi}{2}) \sin\theta\cos\theta] \tag{77b}$$

From Eq. (73a) when n is odd we have

$$\delta^2 J = (-1)^{(n-1)/2}\, \omega^2 A^2 \{[\frac{1}{2} + \frac{n\pi}{2} \sin\theta\cos\theta] + [\frac{1}{2} - \frac{n\pi}{2} \sin\theta\cos\theta]\}$$

$$\delta^2 J = (-1)^{(n-1)/2}\, \omega^2 A^2 \tag{77c}$$

In particular for $n = 1$, one obtains

$$\delta^2 J = \omega^2 A^2 > 0 \tag{78a}$$

which gives a minimum for the functional J. For $n = 3$

$$\delta^2 J = -\omega^2 A^2 < 0 \tag{78b}$$

which gives a maximum for the functional J. It is noted that $\delta^2 J$ is independent of $\theta$ which is related to the starting conditions. It is also independent of the polarity of A since it appears in terms of $A^2$.

Now we take the case when n is even. Substituting Eqs. (74) and (76) into Eq. (73), one obtains

$$\delta^2 J[\delta y_t] = \omega\delta A^2 \int_o^{n\pi/2} \{(\sin\omega t \cos\theta + \cos\omega t \sin\theta)$$

$$[-\sin\omega t\, (-1)^{n/2} \cos\theta + \cos\omega t (-1)^{n/2} \sin\theta]\} d(\omega t)$$

$$= (-1)^{n/2}\, \omega^2 A^2 \int_o^{n\pi/2} [-\sin^2\omega t \cos^2\theta + \cos^2\omega t \sin^2\theta] d(\omega t)$$

$$= (-1)^{n/2}\, \omega^2 A^2 (-\cos^2\theta + \sin^2\theta) n\pi/4 \tag{79a}$$

and

$$\delta^2 J[\omega\delta y] = \omega^2 A^2 \int_0^{n\pi/2} \{(\cos \omega t \cos \theta - \sin \omega t \sin \theta)$$

$$\cdot [\cos \omega t(-1)^{n/2} \cos \theta + \sin \omega t(-1)^{n/2} \sin \theta]\} d(\omega t)$$

$$= (-1)^{n/2} \omega^2 A^2 \int_0^{n\pi/2} [\cos^2 \omega t \cos^2 \theta - \sin^2 \omega t \sin^2 \theta] d(\omega t)$$

$$= (-1)^{n/2} \omega^2 A^2 (\cos^2 \theta - \sin^2 \theta) n\pi/4 \tag{79b}$$

From Eq. (73a) when n is even we have

$$\delta^2 J = (-1)^{n/2} \omega^2 A^2 \{(- \cos^2 \theta + \sin^2 \theta) + (\cos^2 \theta - \sin^2 \theta)\} n\pi/4$$

$$\delta^2 J = 0 \quad \text{for all } n = \text{even} \tag{79c}$$

We can conclude here that the functional J definitely (ref 6) has a minimum if $\omega T = \pi/2$, or T is a quarter of the natural period of the oscillation $\tau = 2\pi/\omega$. Moreover, from Eq. (70) for $n = 1$

$$T = t_b - t_o = \pi/(2\omega) = \tau/4 \tag{80a}$$

If $n = 2$ and $\delta^2 J = 0$ in Eq. (79c), we have

$$T = t_b - t_o < \pi/\omega = \tau/2 \tag{80b}$$

This is the upper limit of the increment we chose for T, above which the minimum of the functional J is not guaranteed.

## EXTREMAL FOR A SIMPLY-SUPPORTED BEAM WITH CONCENTRATED LOAD AT THE MIDDLE

To show that $\delta^2 J$ must be positive semi-definite we use the example of a simply-supported beam with a concentrated load at the middle. If the load is suddenly removed (ref 7), Eq. (63) becomes

$$\eta_{tt} + b^2 \eta_{xxxx} = 0 \tag{81a}$$

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[6]Shen, C. N. and Wu, Julian J., "A New Variational Method for Initial Value Problems, Using Piecewise Hermite Polynomial Spline Functions," ARO Report 81-3, Proceedings of the 1981 Army Numerical Analysis and Computers Conference, 1981.

[7]Jacobsen, Lydkis and Ayre, Robert S., Engineering Vibrations, McGraw-Hill, 1958.

Or from Eq. (56) we have

$$y_{tt} + b^2 y_{xxxx} = 0 \tag{81b}$$

The starting conditions are

$$\frac{d}{dt}\,\sigma_o(x) = 0 \tag{82a}$$

$$\sigma_o(x) = Mc/I = \frac{(Px/2)h}{I} \quad \text{for } 0 < x < \ell/2 \tag{82b}$$

and

$$\sigma_o(x) = \frac{P(\ell/2)h}{I}\left(1 - \frac{x}{\ell}\right) \quad \text{for } \ell/2 < x < \ell \tag{82c}$$

The solution for Eq. (81b) is

$$\sigma(x,t) = -\,Eh\,\frac{\partial^2 y}{\partial x^2}$$

$$= \frac{8\sigma_s}{\pi^2}\sum_{n=odd}^{\infty}\frac{1}{n^2}\,(-1)^{(n-1)/2}\,\sin(n\pi x/\ell)\cos p_n t \tag{83a}$$

where

$$p_n = bn^2\pi^2/\ell^2 \tag{83b}$$

and

$$\sigma_s = (P\ell/2I)(h/2) \tag{83c}$$

The quantity $\sigma_s$ is the initial static stress at the middle of the beam where $x = \ell/2$ and on the outer surface of the beam.

In order to find $y_t$ we let

$$y = \frac{1}{Eh}\,\frac{8\sigma_s}{\pi^2}\sum_{n=odd}^{\infty}\left(\frac{1}{n^2}\right)\left(\frac{\ell}{n\pi}\right)^2(-1)^{\frac{n-1}{2}}\,\sin(n\pi x/\ell)\cos p_n t \tag{84a}$$

Then by partial differentiation we have

$$\frac{\partial^2 y}{\partial x^2} = -\,\frac{1}{Eh}\,\frac{8\sigma_s}{\pi^2}\sum_{n=odd}^{\infty}\left(\frac{1}{n^2}\right)(-1)^{\frac{n-1}{2}}\,\sin(n\pi x/\ell)\cos p_n t \tag{84b}$$

which agrees with Eq. (83a), and

$$\frac{\partial y}{\partial t} = -\,\frac{1}{Eh}\,\frac{8\sigma_s}{\pi^2}\sum_{n=odd}^{\infty}\left(\frac{1}{n^2}\right)(-1)^{\frac{n-1}{2}}\,b\,\sin(n\pi x/\ell)\,\sin p_n t \tag{84c}$$

where from Eq. (83b)

$$b = p_n \ell^2/(n^2\pi^2) \tag{84d}$$

and

$$p_1 = b\pi^2/\ell^2 \tag{84e}$$

It is noted that the index n appears in both spatial and temporal functions in Eqs. (84a) and (84b) under the summation sign. We are interested in finding those functions of t that are independent of the index n. Let us assume that

$$p_n t = \frac{n^2\pi}{\ell} (\ell/2-c)$$

$$= n^2\pi/2 - n^2\pi c/\ell \tag{85}$$

It is noted that for $n = 1, 3,$ and $5$, $n^2\pi/2$ becomes $\pi/2$, $4\pi + \pi/2$, and $12\pi + \pi/2$, respectively.

Thus we have

$$\cos p_n t = \cos [n^2\pi/2 - n^2\pi c/\ell]$$

$$= \cos [\pi/2 - n^2\pi c/\ell]$$

$$= \sin (n^2\pi/c/\ell) \tag{86a}$$

and

$$\sin p_n t = \cos (n^2\pi c/\ell) \tag{86b}$$

Moreover, for $c/(\ell/2) = 1, 1/2,$ and $0$

$$\cos p_n t = \sin[(\pi/2)c/(\ell/2)] = 1, 0.707, \text{ and } 0, \text{ respectively} \tag{87a}$$

and

$$\sin p_n t = \cos[(\pi/2)c/(\ell/2)] = 0, 0.707, \text{ and } 1, \text{ respectively} \tag{87b}$$

The above functions are independent of index n at those values of $c/(\ell/2)$. Thus, Eq. (84) may be rewritten at those values as

$$\frac{\partial^2 y}{\partial x^2} (x,t=\hat{t}) = - \frac{\sigma_s}{Eh} \sin(\pi c/\ell) y_o(x) \tag{88a}$$

and

$$\frac{\partial y}{\partial t}(x, t = \hat{t}) = -\frac{\sigma_s}{Eh}\cos(\pi c/\ell) y_o(x) \quad (88b)$$

where

$$y_o(x) = \sum_{n=odd}^{\infty} (-1)^{\frac{n-1}{2}} \sin(n\pi x/\ell) \quad (88c)$$

The series terms in Eq. (88c) are the result of an expansion of a triangular deflection of the form

$$y_o(x) = x(\ell/2) \qquad \text{for } 0 < x < \ell/2 \quad (88d)$$

$$y_o(x) = 2 - x/(\ell-2) \qquad \text{for } \ell/2 < x < \ell \quad (88e)$$

as shown in Figure 2.

For the images of Eqs. (84b) and (84c) the time dependent terms become

$$\cos p_n t(T-t) = \cos(p_n T - p_n t) \quad (89a)$$

and

$$\sin p_n(T-t) = \sin(p_n T - p_n t) \quad (89b)$$

The term $p_n T$ can be obtained from Eqs. (84d) and (84e) as

$$p_n T = (bn^2\pi^2/\ell^2)T = n^2 p_1 T \quad (89c)$$

For computation by the finite element method the increment in time is taken as T which is defined as

$$T \stackrel{\Delta}{=} t_b - t_o = (m/p_1)(\pi/2) \quad (90a)$$

where

$$m = 1,2,3... \quad (90b)$$

Then with the aid of Eqs. (85), (89c), and (90a) we have

$$p_n(T-t) = mn^2(\pi/2) - [n^2\pi/2 - n^2\pi c/\ell]$$

$$= (m-1)n^2(\pi/2) + n^2 c/\ell \quad (90c)$$

Then for the case when m = 1, the time dependent terms become

$$\cos p_n(T-t) = \cos(n^2\pi c/\ell) \quad (91a)$$

and

$$\sin p_n(T-t) = \sin(n^2 \pi c/\ell) \tag{91b}$$

By similar method we can obtain

$$\frac{\partial^2 y}{\partial x^2}(x, \hat{t}=T-t) = -\frac{\sigma_s}{Eh}\cos(\pi c/\ell) y_o(x) \tag{92a}$$

$$\frac{\partial y}{\partial t}(x, \hat{t}=T-t) = -\frac{b\sigma_s}{Eh}\sin(\pi c/\ell) y_o(x) \tag{92b}$$

For the partial differential equation we have the second variation from Eq. (52) which gives

$$\delta^2 J = \int_o^T [\delta y_t(x, \hat{t}=t)\delta y_t(x, \hat{t}=T-t)$$

$$+ b^2 \delta y_{xx}(x, \hat{t}=t)\delta y_{xx}(x, \hat{t}=T-t)]dt \tag{93}$$

Separating Eq. (93) into two parts and using Eqs. (88) and (92), we have

$$\delta^2 J = \delta^2 J[\delta y_t] + \delta^2 J[b\delta y_{xx}] \tag{94}$$

The first term on the right of Eq. (94) is

$$\delta^2 J[\delta y_t] \cong \int_{x_o}^{x_b} (b\sigma_s/Eh)^2 y_o^2(x)dx \int_o^{p_1 T=\pi/2} \cos(\pi c/\ell)\sin(\pi c/\ell)d(p_1 t)$$

$$\cong \int_{x_o}^{x_b} (b\sigma_s/Eh)^2 y_o^2(x)dx > 0 \tag{95a}$$

where

$$p_1 t = \pi/2 - \pi c/\ell \qquad d(p_1 t) = -(\pi/\ell)dc \tag{95b}$$

at

$$p_1 t = \pi/2 \quad , \quad c/(\ell/2) = 0 \quad , \quad \pi c/\ell = 0 \tag{95c}$$

at

$$p_1 t = 0 \quad , \quad c/(\ell/2) = 1 \quad , \quad \pi c/\ell = \pi/2 \tag{95d}$$

The second term of Eq. (94) is

$$\delta^2 J[b\delta y_{xx}] = \int_{x_o}^{x_b} (b\sigma_s/Eh)^2 y_o^2(x)dx \int_o^{\pi/2} \sin(\pi c/\ell)\cos(\pi c/\ell)d(\pi c/\ell)$$

$$= \int_{x_o}^{x_b} (b\sigma_s/Eh)^2 y_o^2(x)dx > 0 \tag{95e}$$

Thus by combining Eqs. (95a) and (95e), one obtains

$$\delta^2 J = \int_{x_o}^{x_2} 2(b\sigma_s/Eh)^2 y_o^2(x)dx > 0 \qquad (96)$$

which gives a minimum for the functional J.

Now we take the case when m = 2, Then Eqs. (90c) and (89) become

$$p_n(T-t) = n^2\pi/2 + n^2\pi c/\ell \qquad (97a)$$

$$\cos p_n(T-t) = \cos(n^2\pi/2+n^2\pi c/\ell) \quad , \quad n^2 = 1,9,25, \text{ etc.}$$

$$= -\sin(n^2\pi c/\ell) \qquad (97b)$$

and

$$\sin p_n(T-t) = \sin(n^2\pi/2+n^2\pi c/\ell) \quad , \quad n^2 = 1,9,25,\text{etc.} \qquad (97c)$$

$$= \cos(n^2\pi c/\ell)$$

Thus the image function becomes

$$\frac{\partial^2 y}{\partial x^2}(x,\hat{t}=T-t) = -\frac{\sigma_s}{Eh}(-\sin \pi c/\ell)y_o(x) \qquad (98a)$$

$$\frac{\partial y}{\partial t}(x,\hat{t}=T-t) = -\frac{\sigma_s}{Eh}\cos(\pi c/\ell)y_o(x) \qquad (98b)$$

By substituting Eqs. (88) and (98) into Eq. (94) we have

$$\delta^2 J \cong \int_{x_o}^{x_b}(b\sigma_s/Eh)^2 y_o^2(x)dx\int_{o}^{\pi/2}[\cos^2(\pi c/\ell) - \sin^2(\pi c/\ell)]d(\pi c/\ell)$$

$$\cong \int_{x_o}^{x_b}(b\sigma_s/Eh)^2 y_o^2(x)dx(\frac{1}{2})[\sin \pi - \sin 0) = 0 \qquad (99a)$$

We can conclude here that the functional J definitely (ref 8) has a minimum if $p_1 T = \pi/2$, where T is a quarter of the natural period of the oscillation $\tau = 2\pi/p_1$. Moreover, from Eq. (90a) for m = 1, we have

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[8]Shen, C. N., "Variational Principle for Gun Dynamics With Adjoint Variable Formulation," Proceedings of the Third US Army Symposium on Gun Dynamics, Volume II, May 1982, p. IV-108.

$$T = t_b - t_o = \pi/(2p_1) = \tau/4 \tag{99b}$$

If $m = 2$ and $\delta^2 J = 0$ in Eq. (99a), we can conclude that

$$T = t_b - t_o < \pi/p_1 = \tau/2 \tag{99c}$$

This is the upper limit of the increment we choose for T, above which the minimum of the functional J is not guaranteed.

CONCLUSIONS

The functional in bilinear form is symmetrical about the original variables and the adjoint variables. The Euler-Lagrange equations for the original and the adjoint systems are derived using the fundamental lemma of the calculus of variations. By integrating the bilinear expression by parts, one can obtain the bilinear concomitant in terms of initial and boundary terms. The adjoint system can be arranged in a manner that it is a reflected mirror of the original system in time. Thus the initial conditions for the bilinear concomitant become zero.

Generalized boundary conditions using many types of "springs" relating the various spatial partial derivatives are defined to satisfy the boundaries of the concomitant. The higher partials in original variables and variations in the adjoint variables can be kept in low orders by these "springs". Algorithms are developed for use in the finite element method by taking the first variations of the functional. These algorithms are simplified because the adjoint system gives exactly the same solutions as that of the original system.

Sensitivity coefficient is found to be similar to the variation of the variable and obeys the same partial differential equation. The solution of

the original PDE is taken as the solution of the variations for two examples, a simple oscillator and a simply-supported beam with load at the middle. It is found that the second variation of the functional is positive semi-definite if the increment in time for the finite element method is less than half the natural period of the physic systems in both cases. This will guarantee a minimum for the functional and thus the method is truly workable if employed as algorithms for the finite element method.

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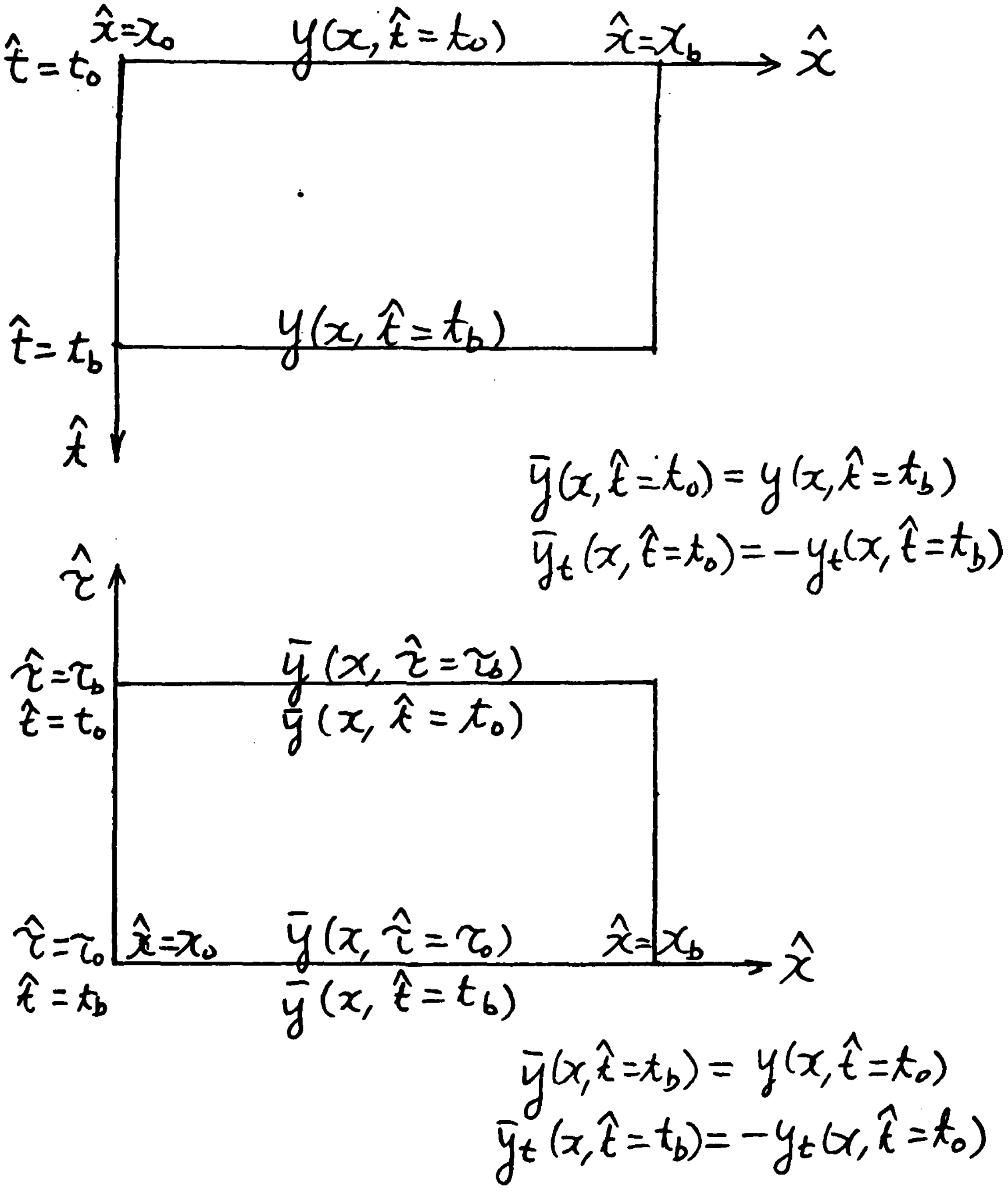


**Figure 1. Image Reflection of the Adjoint System.**

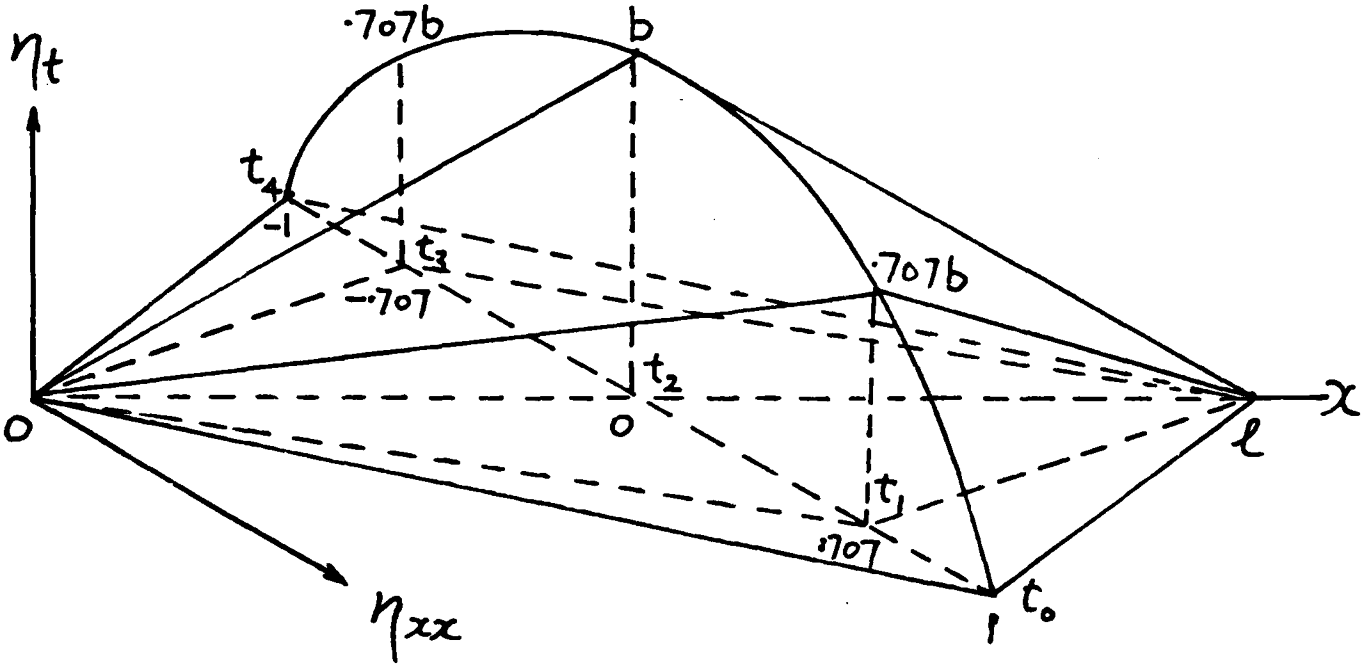


Figure 2. Variation of the Partials for a Beam Equation.

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